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# Exam II, Abstract Algebra, MTH 320,Fall 2017

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Score = $\frac{\quad}{63}$

47/47

**QUESTION 1.** Let $(D, *)$ be a finite group with 245 elements. Assume that $D$ has a normal subgroup with 5 elements and it has also a subgroup with 49 elements. Prove that $D$ is an abelian group. Up to isomorphism, find all possible structures of $D$.

$|D| = 245$. $\exists H_1 \lhd D$ st $|H_1| = 5$ and $\exists H_2 \lhd D$ s.t. $|H_2| = 49$.

Why!

To Prove: D is Abelian.

7/10

$H_1 * H_2 \lhd D$. $|H_1 * H_2| = \frac{|H_1||H_2|}{|H_1 \cap H_2|}$

$\therefore |H_1 * H_2| = \frac{|H_1||H_2|}{1} = \underline{245}$. $\therefore H_1 * H_2 = D$.

But $|H_1 \cap H_2| = 1$.
$\because H_1 \cap H_2 = \{e\}$ (or) $H_1$
($\because |H_1|$ is prime). But $|H_1| \nmid 49$
So $H_1 \cap H_2 = \{e\}$

Further: $H_1 \cap H_2 = \{e\}$ (Explained $\rightarrow$).

$\therefore D \cong H_1 \times H_2$. $|H_1| = 5 \Rightarrow$ Abelian. $|H_2| = 49 = p^2$ $(p = 7)$ $\therefore$ Abelian

$\therefore H_1 \times H_2$ is Abelian $\Rightarrow$ D is Abelian.

$H_1 \cong \mathbb{Z}_5$ and $H_2 \cong \mathbb{Z}_{49}$ (or) $\mathbb{Z}_7 \times \mathbb{Z}_7$ [Classification of Abelian groups]

$\therefore$ $D \cong \mathbb{Z}_5 \times \mathbb{Z}_{49}$ (or) $D \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_7$

**QUESTION 2.** Let $(D, *)$ be a finite group with 125 elements. Prove that $D$ is not simple.

$|D| = 125$ is a finite group

$\therefore |D| = p^3$. $\therefore |C(D)| \geq p$ (i.e. $\geq 5$).

5/5

$\therefore \exists H = C(D) \lhd D$.

But the centre is always a Normal Group.

$\therefore |C(D)| \geq p$ and $\underline{C(D) \lhd D}$.

If $|C(D)| = 5$ (or) 25, $\exists H$ st $|H| = 5$ (or) 25 s.t $H \lhd D$.

If $|C(D)| = 125$, the group is Abelian (PTO)

But

converse of Lagrange Theorem is True for Abelian Groups.

$\therefore \exists\ H_1, H_2$ st $|H_1| = 5$, $|H_2| = 25$

and $H_1 \lhd D$, $H_2 \lhd D$

(All Subgroups of Abelian Groups are Normal)

$\therefore$ In All Cases,

we have normal Subgroups in $D$ which are non-trivial, and not Equal to $D$

$\therefore D$ is never Simple.

**QUESTION 3.** Does $A_6$ have a subgroup, say $H$, of order 72? if yes, then what is the maximal order of a cyclic subgroup of $H$. If No, then explain clearly.

~~$|A_6| = 360$ A has elements of order 2, 3, 5 by Cauchy.~~
~~'5' is the maximum possible order~~
~~If H had a S.G. of order 72, the maximal cyclic subgroup of H would have~~

$A_6$ is Simple. If $A_6$ had S.G of order 72, then $[A_6 : H] = 5$.

$\therefore \exists f : A_6 \longrightarrow S_5$ which is a non-trivial homomorphism

6/6

$Ker(f) \neq A_6$. $Ker(f) \neq \{e\}$ $\because$ $A_6/Ker(f) \cong$ Range(f) and if $Ker(f)$

$= \{e\}$ then $A_6/\{e\} \cong L$, where $L < S_5$

But $\dfrac{|A_6|}{|\{e\}|} = 360$ and $|S_5| = 120$ (Impossible for Subgroup to have more Elements than Group).

$\therefore$ $Ker(f) \neq \{e\} \neq A_6$ and $Ker(f) \lhd A_6$. But $A_6$ is Simple.
Contradiction

**QUESTION 4.** (i) Is $Z_2 \times Z_4 \times Z_{12}$ isomorphic to $Z_8 \times Z_{12}$? EXPLAIN

NO. Deny. Then $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{12} \cong \mathbb{Z}_8 \times \mathbb{Z}_{12}$

$\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \cong \mathbb{Z}_8$.

5/5

But, $\exists$ $a \in \mathbb{Z}_8$ st $|a| = 8$ but not in $\mathbb{Z}_2 \times \mathbb{Z}_4$.

contradiction

(ii) Let $n = 2^7 \cdot 5^2 \cdot 7^3$. Write $U(n)$ in terms of products of its invariant factors.

$n = 2^7 * 5^2 * 7^3$

$\therefore U(n) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^5} \times \mathbb{Z}_{20} \times \mathbb{Z}_{294}$

i.e. $\mathbb{Z}_2 \times \mathbb{Z}_{32} \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{49}$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{23520}$

5/5

(iii) Let $F$ be an abelian group with $3^4 \cdot 11^2$ elements. Up to isomorphism, find all possible structures of $F$. Partition:

$\therefore F \cong \mathbb{Z}_{3^4} \times \mathbb{Z}_{11^2}$ (OR) $\mathbb{Z}_{3^4} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$

(OR) $\mathbb{Z}_{3^3} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{11^2}$ (OR) $\mathbb{Z}_{3^3} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$

(OR) $\mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{11^2}$ (OR) $\mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$

(OR) $\mathbb{Z}_{3^2} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{11^2}$ (OR) $\mathbb{Z}_{3^2} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$

(OR) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11^2}$ (OR) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$.

| 4 | 2 |
|---|---|
| 4 | 2 |
| 3+1 | 1+1 |
| 2+2 | |
| 1+1+2 | |
| 1+1+1+1 | |

10/10

(iv) Let $F$ be an abelian group with $5^3 \cdot 7$ elements. Assume $F$ has a unique subgroup with 25 elements. Up to isomorphism, find all possible structures of $F$.

Without Constraint: $\mathbb{Z}_{5^3} \times \mathbb{Z}_7$ (OR) $\mathbb{Z}_{5^2} \times \mathbb{Z}_5 \times \mathbb{Z}_7$ (OR) $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_7$

$\mathbb{Z}_{5^3} \times \mathbb{Z}_7$ has Unique Subgroup with 25 Elements.

but others have more than 1 Subgroup with 25 Elements

$\therefore F \cong \mathbb{Z}_{5^3} \times \mathbb{Z}_7$ //.

Partitions

| 3 | 1 |
|---|---|
| 3 | 1 |
| 2+1 | |
| 1+1+1 | |

5/5

**QUESTION 5.** (Bonus) Assume that $D$ is a group with $3^{2017} \cdot 5^2$ elements. Assume that $D$ has a unique subgroup, say $H$ with 3 elements and also assume that $D/H$ is a cyclic group. Prove that $D$ is a cyclic group. . ~~Assume that $H$ is a normal subgroup of $D$ such that $H$ has.~~

3/3

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Ans) $D = 3^{2017} * 5^2$. Let $p = 3$, $i = 5^2$.

i.e. $D = p^n i$ and $\gcd(p,i) = \gcd(3,5^2) = 1$.

$D$ has Unique Subgroup, $H$ st $|H| = 3$.

$D/H$ is Cyclic.

$\therefore D/H = \langle a * H \rangle$ for some $a \in D$.

Consider: $f : D \longrightarrow D$ st $f(d) = d^p$.

This is clearly homomorphism.

$Ker(f) = H$ $\quad (\because d^p = e \Rightarrow |d| = p \quad \because p$ is prime$)$.

$D/Ker \cong Range \Rightarrow |D/H| = \frac{|D|}{|H|} = \frac{p^n i}{p} = p^{(n-1)} i$.

$|Range(f)| \mid |D| \Rightarrow p^{(n-1)} i \mid p^n i$

(PTO)

$\therefore |D| = p^{(n-1)} i$ (OR) $|D| = p^n i$.

We show that $|D| = p^n i$.

In both cases $\Rightarrow \exists$ Unique Subgroup $K$ in $D$ of order $p$. $\therefore \underline{K = H}$.

But this $K$ is made of powers of a

$$\therefore \quad H = \{a^{i_1}, a^{i_2}, \ldots, a^{i_s}\}.$$

For any $d \in D$

$$d * H = a^m * H$$

$$\Downarrow$$

$$d = a^m * h$$

$$= a^m * a^{i_k} \quad \text{for some } i_k$$

$$d = a^{m+i_k} \Rightarrow \underline{d = a^x} \qquad (x = m + i_k)$$

$\therefore D$ is Cyclic.

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